THE LINEARIZABLE QAP AND SOME APPLICATIONS IN OPTIMIZATION PROBLEMS IN GRAPHS

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joint work with

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Books and surveys: Burkard et al. 1998, Ç. 1998, Loyola et al. 2007

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- polynomially solvable special cases for specially structured coefficient matrices A and B
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Equivalent formulation as $QAP(A, B)$ of size n:
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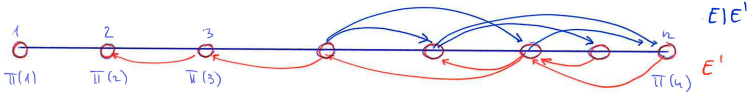
For any ordering π of the vertices of G: $\sum_{i,j=1}^{n} a_{\pi(i)\pi(j)} b_{ij}$ is the number of arcs leading from vertices with order $i \geq 2$ to vertices with order $j < i$; these arcs build a feedback arc set.

The FAS-QAP

$G = (V, E \setminus E')$ is acyclic $\iff$ $\exists$ linear ordering $\pi : V \to \{1, 2, \dots, n := |V|\}$ of the vertices with $(\pi(i), \pi(j)) \in E \setminus E' \Rightarrow i < j$



Let $V = \{1, 2, \dots, n\}$

$\iff$ $\exists$ permutation π of $\{1, 2, \dots, n\}$ with $i > j \Rightarrow (\pi(i), \pi(j)) \notin E \setminus E'$

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$$\min_{\pi} |E'| \iff \min_{\pi} \sum_{i,j=1}^{n} a_{\pi(i)\pi(j)} b_{ij}$$

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A $QAP(A, B)$ of size n is called *linearizable* if there exists an $n \times n$ matrix C such that

$$\sum_{ij=1}^{n} a_{\pi(i)\pi(j)} b_{ij} = \sum_{i=1}^{n} c_{i\pi(i)} \text{ for all permutations } \pi \text{ of } \{1, 2, \ldots, n\}$$

(Bookhold, 1990)

Which special cases of the $QAP(A, B)$ are linearizable?

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A is a *weak sum matrix* iff it can be turned into a sum matrix by appropriately changing its diagonal elements.

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$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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If $QAP(A_1,B)$ and $QAP(A_2,B)$ are linearizable, then $QAP(\lambda_1 A_1 + \lambda_2 A_2, B)$ is also linearizable for any two reals λ_1, λ_2.

Some definitions

A $n \times n$ matrix $A = (a_{ij})$ is called a *directed cut matrix*, iff there exists a subset of indices $\emptyset \neq I \leq \{1, 2, \ldots, n\}$, such that $a_{ij} = 1$ for $i \in I$ and $j \notin I$ and $a_{ij} = 0$ otherwise.

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Three indices i, j, k are said to form a *balanced 3-cycle* in an $n \times n$ matrix A, if the corresponding entries satisfy

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Theorem 1: An $n \times n$ matrix A is a balanced 3-cycle matrix iff it can be written as the sum of a symmetric matrix and a linear combination of directed cut matrices.

The linearizable FAS-QAP

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Then

$$Z(A, B, \pi) = \sum_{\substack{i,j=1 \\ i>j}}^{n} a_{\pi(i)\pi(j)} = \sum_{i=1}^{k} (p_i - i) = \sum_{i=1}^{n} c_{i\pi(i)}$$

with $c_{ij} = 1$ if $i = 1, 2, \ldots, k$, $j \in \{1, 2, \ldots, n\}$ and $c_{ij} = 0$ otherwise.

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Theorem 2: The FAS-QAP with a coefficient matrix A (and B a 0-1 lower triangular matrix with 1-s below the main diagonal and 0-s above it) is linearizable iff A is a balanced 3-cycle matrix.

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Theorem The TSP-QAP for matrix A is linearizable, if and only if A is a weak sum matrix.

Summary and outlook

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- Complete combinatorial characterization of all linearizable asymmetric QAP instances. (ambicious goal!)

THANK YOU!